Applications of Hamilton's equations of motion :

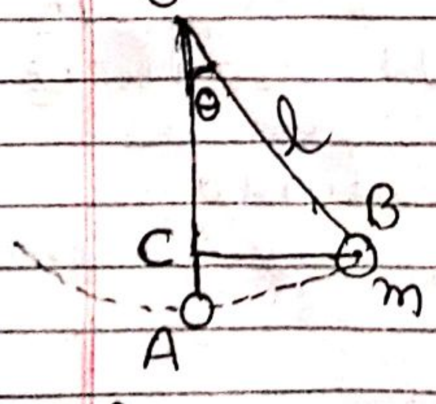
(1) Simple pendulum:

If the string is of length l, then Kinetic energy is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\theta})^2$$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

Where, m is the mass of the bob.



In coming from position B to A, the mass has fallen freely through a vertical distance CA. Thus potential energy is

$$V = mg(OA - OC)$$

$$= mg(l - l\cos\theta)$$
$$= mgl(1-\cos\theta)$$

~~Where~~ So that Lagrangian is,

$$L = T - V$$
$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

giving, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

The Hamiltonian is,

$$H = \sum p_j \dot{q}_j - L$$
$$= p_\theta \cdot \dot{\theta} - L$$
$$= ml^2\dot{\theta}^2 - \left\{\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)\right\}$$
$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$H = T + V$$

which implies that system is Conservative.

Now putting eq^n 2 into eq. 3, we get

$$H = \frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 + mgl(1-\cos\theta)$$

Giving,

$$\left.\begin{aligned} \frac{\partial H}{\partial p_\theta} &= \frac{p_\theta}{ml^2} \\ \frac{\partial H}{\partial \theta} &= mgl\sin\theta \end{aligned}\right] \quad \text{--- (4)}$$

Thus Hamilton's equations of motion for this system will be

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \quad \text{--- (5)}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \quad \text{--- (6)}$$

Putting the value of $\dot{p}_\theta$ from eq. (6) into eq. (5), we find

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

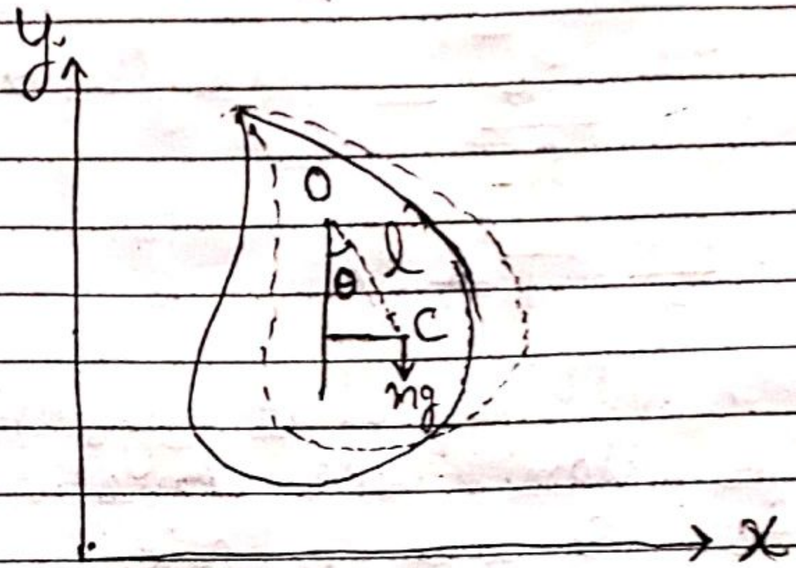
or, $$\ddot{\theta} + \frac{g}{l}\theta = 0, \quad \left[\text{for small } \theta, \sin\theta = \theta\right]$$

which is the equation of motion of a simple pendulum with period $2\pi\sqrt{\frac{l}{g}}$.

(2) <u>Compound Pendulum:</u>

If θ be the angle through which the body is deflected, then kinetic energy is

$$T = \frac{1}{2} I \dot{\theta}^2$$



The potential energy relative to a horizontal plane through O is

$$V = -mgl \cos\theta$$

giving,

$$L = \frac{1}{2} I \dot{\theta}^2 + mgl \cos\theta$$

Then, $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \quad \text{——— (1)}$$

The Hamiltonian is then,

$$= \sum p_j \dot{q}_j - L$$

$$= p_\theta \cdot \dot{\theta} - L$$

$$= I\dot{\theta}.\dot{\theta} - \frac{1}{2} I \dot{\theta}^2 - mgl \cos\theta$$

$$= \frac{1}{2} I \dot{\theta}^2 - mgl \cos\theta$$

$$= \frac{1}{2} I \left(\frac{p_\theta}{I} \right)^2 - mgl \cos\theta$$

$$= \frac{p_\theta^2}{2I} - mgl \cos\theta \quad \text{——— (2)}$$

$$= T + V$$

Which implies that system is conservative.
The Hamilton's equations of motion for θ and p_θ are

$$\left.\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \end{aligned}\right] \quad \text{——— (3)}$$

From equation (2), we find

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}, \quad \frac{\partial H}{\partial \theta} = mgl \sin\theta$$

Putting these in eqs. (3), we get

$$\dot{\theta} = \frac{p_\theta}{I}$$

and, $$\dot{p}_\theta = -mgl \sin\theta$$

Putting value of p_θ from eq. (4) into eq (5) we get

$$I \ddot{\theta} = -mgl \sin\theta$$

or, $\ddot{\theta} + \frac{mgl}{I}\theta = 0$

[for small θ, $\sin\theta \simeq \theta$]

Which is the equation of motion of Compound pendulum, with $\omega = \sqrt{\frac{mgl}{I}}$